On vertices of given degree in Polya trees

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Let $T_n$ be the set of rooted unlabelled non-plane trees (Polya trees) on $n$ vertices. It has been shown that the random variable $X^{(d)}_n$, counting the number of vertices of degree $d$ in a tree $T \in T_n$ drawn uniformly at random, has expected value $E(X^{(d)}_n) = \mu_d n + O(1)$, with $\mu_d = \frac{2^d}{\sqrt{\rho}} \rho^d$, where $\rho \approx 0.3383219$ is the singularity of the generating function $T(x)$ of Polya trees and $C \approx 7.7581604$ and $b \approx 2.6811266$ are known constants.

Let $L^{(d)}_n(k)$ be the number of nodes of degree $d$ at distance $k$ from the root in a random Polya tree $T \in T_n$ of size $n$, and $L^{(d)}_n(t)$ be the stochastic process obtained by linear interpolation. We prove the following refinement.

**Theorem 1.** Let $l^{(d)}_n(t) = \frac{1}{\sqrt{n}} L^{(d)}_n(t \sqrt{n})$, and $l(t)$ denote the local time of a standard Brownian excursion. Then $l^{(d)}_n(t)$ converges weakly to the local time of a Brownian excursion, i.e., we have

$$(l^{(d)}_n(t))_{t \geq 0} \xrightarrow{w} l\left(\frac{b \sqrt{\rho}}{2 \sqrt{2}} \cdot \left(\frac{b \sqrt{\rho}}{2 \sqrt{2}}\right)_{t \geq 0}\right).$$

We further compute the correlation of two different degrees $d_1$ and $d_2$ on a given level $k$ and prove that the correlation coefficient is 1, asymptotically as $n$ tends to infinity.