Coloring uniform hypergraphs with bounded edge degree

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Let H be a hypergraph and let $\Delta_e(H)$ denote the maximum edge degree of H. In 1973 P. Erdős and L. Lovász (see [1]) stated the following problem: find the value $\Delta_e(n,r)$ equal to the minimum possible $\Delta_e(H)$, where H is an n-uniform non-r-colorable hypergraph. By using Local Lemma they proved that

$$\Delta_e(n,r) \geq \frac{1}{e} r^{n-1}. \tag{1}$$

This bound was improved by J. Radhakrishnan and A. Srinivasan in 2000 in the case of two colors. They showed (see [2]) that, for sufficiently large n,

$$\Delta_e(n,2) \geq 0.17 \left(\frac{n}{\ln n}\right)^{\frac{1}{2}} 2^n.$$

In our work we improve the classical result (1) of Erdős and Lovász as follows.

Theorem 1. *For every $n \geq 3$, $r \geq 3$, the following inequality holds*

$$\Delta_e(n,r) \geq \frac{1}{8}\sqrt{n}\, r^{n-1}.$$

We also study the value $\Delta_e(n,r,s)$ equal to the minimum possible $\Delta_e(H)$, where H is an n-uniform non-r-colorable hypergraph with girth at least $s+1$. It is clear that $\Delta_e(n,r,1) = \Delta_e(n,r)$ and $\Delta_e(n,r,s) \leq \Delta_e(n,r,s+1)$. Erdős and Lovász (see [1]) showed that, for all $s \geq 1$,

$$\Delta_e(n,r,s) \leq 20\, n^3\, r^{n+1}.$$

This upper bound was improved by A.V. Kostochka and V. Rödl (see [3]):

$$\Delta_e(n,r,s) \leq n^2\, r^{n-1} \ln r. \tag{2}$$

Our second main result gives a new lower bound for the value $\Delta_e(n,r,3)$.

Theorem 2. *There exists an integer n_0 such that, for every $n \geq n_0$ and every $r \geq 2$, the following inequality holds*

$$\Delta_e(n,r,3) \geq r^{n-1} n^{1-4\left\lfloor\sqrt{\frac{\ln n}{\ln(2\ln n)}}\right\rfloor^{-1}}. \tag{3}$$

Our bound (3) asymptotically improves all previously known results. It is easy to see that the upper bound (2) is only $n^{1+o(1)} \ln r$ times greater than (3).

The proofs of Theorem 1 and Theorem 2 are based on two different modifications of the random recoloring method.

References

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