Coloring uniform hypergraphs with bounded edge degree

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Let $H$ be a hypergraph and let $\Delta_e(H)$ denote the maximum edge degree of $H$. In 1973 P. Erdős and L. Lovász (see [1]) stated the following problem: find the value $\Delta_e(n, r)$ equal to the minimum possible $\Delta_e(H)$, where $H$ is an $n$-uniform non-$r$-colorable hypergraph. By using Local Lemma they proved that

$$\Delta_e(n, r) \geq \frac{1}{e} r^{n-1}. \quad (1)$$

This bound was improved by J. Radhakrishnan and A. Srinivasan in 2000 in the case of two colors. They showed (see [2]) that, for sufficiently large $n$,

$$\Delta_e(n, 2) \geq 0.17 \left( \frac{n}{\ln n} \right)^{\frac{1}{2}} 2^n.$$

In our work we improve the classical result (1) of Erdős and Lovász as follows.

**Theorem 1.** For every $n \geq 3$, $r \geq 3$, the following inequality holds

$$\Delta_e(n, r) \geq \frac{1}{8} \sqrt{n} r^{n-1}.$$  

We also study the value $\Delta_e(n, r, s)$ equal to the minimum possible $\Delta_e(H)$, where $H$ is an $n$-uniform non-$r$-colorable hypergraph with girth at least $s + 1$. It is clear that $\Delta_e(n, r, 1) = \Delta_e(n, r)$ and $\Delta_e(n, r, s) \leq \Delta_e(n, r, s + 1)$. Erdős and Lovász (see [1]) showed that, for all $s \geq 1$,

$$\Delta_e(n, r, s) \leq 20 n^3 r^{n+1}.$$

This upper bound was improved by A.V. Kostochka and V. Rödl (see [3]):

$$\Delta_e(n, r, s) \leq n^2 r^{n-1} \ln r. \quad (2)$$

Our second main result gives a new lower bound for the value $\Delta_e(n, r, 3)$.

**Theorem 2.** There exists an integer $n_0$ such that, for every $n \geq n_0$ and every $r \geq 2$, the following inequality holds

$$\Delta_e(n, r, 3) \geq r^{n-1} n^{1-4 \sqrt{\frac{\ln n}{\ln (r \ln n)}}}^{-1}. \quad (3)$$

Our bound (3) asymptotically improves all previously known results. It is easy to see that the upper bound (2) is only $n^{1+o(1)} \ln r$ times greater than (3).

The proofs of Theorem 1 and Theorem 2 are based on two different modifications of the random recoloring method.

**References**


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